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Y. Zhou, W. H. Matthaeus

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Ye Zhou

Lawrence Livermore National Laboratory

University of California, Livermore, California 94511

W. H. Matthaeus

Bartol Research Institute, University of Delaware, Newark, Delaware 19716

Magnetohydrodynamics (MHD) turbulence theory, often employed satisfactorily in astrophysical applications, has often focused on parameter ranges that imply nearly equal values of kinetic and magnetic energies and length scales. However, MHD flow may have disparity magnetic Prandtl number, dissimilar kinetic and magnetic Reynolds number, different kinetic and magnetic outer length scales, and strong anisotropy. Here a phenomenology for such "non-equipartitioned" MHD flow is discussed. Two conditions are proposed for a MHD flow to transition to strong turbulent flow, extensions of (i) Taylor's constant flux in an inertial range, and (ii) Kolmogorov's scale separation between the large and small scale boundaries of an inertial range. For this analysis, the detailed information on turbulence structure is not needed. These two conditions for MHD transition are expected to provide consistent predictions and should be applicable to anisotropic MHD flows, after the length scales are replaced by their corresponding perpendicular components. Second, it is stressed that that the dynamics and anisotropy of MHD fluctuations is controlled by the relative strength between the straining effects between eddies of similar size and the sweeping action by the large-eddies, or propagation effect of the large-scale magnetic fields, on the small scales, and analysis of this balance in principle also requires consideration of non-equipartition effects.

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I. INTRODUCTION

A magnetofluid model can be applied to any fluid or plasma that conducts electric current reasonable well. Described by magnetohydrodynamics (MHD) [1] [2] [3], such system can be found in astrophysical settings; nearby geophysical settings, and laboratory settings. There are many instance of MHD turbulence which has been applied to studying the Sun, the solar wind, the interstellar medium, galaxy clusters, accretion disks, Jupiter, and molecular clouds.

For example, to produce the hot (10^6 K) solar corona, a fraction of the kinetic energy in the Sun's internal convective motions must be converted into heat and transported above the photosphere [4]. Despite more than half a century of research, the precise physical process that leads to corona heating and the subsequent acceleration of the solar wind are not known. However, the list of possible physical processes has narrowed significantly by the advance in measurements such as made by the Ultraviolet Coronagraph Spectrometer aboard SOHO. One surviving possibility is that turbulent cascade driven by low-frequency MHD waves in the photosphere produces strong turbulent heating in the corona.

Another important case is turbulence observed in the interplanetary medium [5] [6] [7], which shows, for example, a near- power law for some two decades in wavenumber [8] with spectral slope -1.73 ± 0.08 and almost equi-partititioned magnetic and kinetic energy in the observed inertial range.

In general if one wants to identify candidate for MHD turbulence, the kinetic and magnetic Reynolds number, Re and Rm, are useful parameters. These indicate the relative importance of the flow's inertial forces to the viscous or magnetic diffusivity effects.

The kinetic Reynolds number, Re, is defined as

$$Re = \tilde{u} L_u / \nu \tag{1}$$

where $\tilde{u}$ is a typical flow velocity (the root-mean-square, r.m.s., of the fluctuating velocity field), ν is the kinematic viscosity (molecular viscosity/density) and L_u a typical (large) velocity scale. Similarly, Rm is the magnetic Reynolds number

$$Rm = \tilde{u}L_u/\mu \tag{2}$$

where μ is the magnetic diffusivity. Here we assume the transport coefficients, ν and μ, uniform.

In many cases, it is assumed, only sometime with observational motivation, that turbulent flow $\tilde{u}$ is on the same order as the typical magnetic filed strength $\tilde{B}$. Such an assumption, often called "equipartition of energy," greatly simplifies the analysis and receives support from solar wind observation. The Alfvén ratio

$$r_A = E_u/E_B \tag{3}$$

is order one, and usually $\approx 1/2$ in the inertial range [8,6]. Similarly, equating kinetic outer scale L_u and magnetic outer scale L_B is often assumed. Here we use the term "non-equipartition" to refer to cases in which $E_u \neq E_B$ or $L_u \neq L_B$ or both. We also include in this designation cases of unequal dissipation scales, which will be defined below. For many astrophysical applications, the assumptions of equipartition are questionable or even disallowed.

A dissimilarity of magnetic and kinetic Reynolds number is one indication of possible non-equipartition, or possible departure from symmetry, between flow and magnetic filed in MHD turbulence. For example in the environments such as the interstellar medium and protogalactic plasma, the magnetic Prandtl number

$$Pr = \frac{Rm}{Re} = \frac{\nu}{\mu} \tag{4}$$

is very large and can be as large as 10^{14} - 10^{22} (Schekochihin *et al.* [9] and West *et al.* [10]). The Galaxy Re is in the range of 10^4 -10^6 and the magnetic Reynolds number is about 10^{19} – 10^{20} [11] .

The huge magnetic Reynolds number in the astrophysical and cosmological situation is often taken to mean that the magnetic field is frozen into plasma, and the scale length of the field increases only with the expansion of the Universe. As pointed out by Christensson *et al.* [12], however, the simply picture does not necessarily give a full description of the dynamics because the MHD turbulence, especially in high Reynolds numbers where nonlinear terms are important, exhibit turbulent behavior, which can leads to a redistribution of magnetic energy over different length scale. Statistical closure theory of Pouquet *et al.* [13] and Leorat *et al.* [14] has been applied to show that decaying MHD turbulence typically leads to a faster growth of the magnetic correlation length than one would expected from the expansion of the Universe along [15].

Yousef *et al.* [16] discussed the situation when the Pr number is very small (10^{-5}). This has dramatic consequence for the magnetorotational instability. This instability is generally accepted as the main mechanism producing turbulence in accretion discs (Balbus and Hawley [17]). When the Pr is sufficient small, however, this instability is suppressed. On the other hand, the Reynolds number is quite large (10^5 or 10^6).

Compressible MHD has been argued as relevant to the star formation where the turbulent Mach number may be higher than 15. We will not discuss this important topic but refer the reader to a recent review (Mac Low and Klessen [18]).

II. MHD EQUATIONS

Scale separation is often employed for simplify the analysis [19]. Specifically, one divides the dynamics into a small-scale part that contains small-small and large-small couplings, and the large-scale part that includes large-large and large-small coupling. When the small scale fields are broadband, one tends to treat the small-small coupling as turbulent and can be approximated as incompressible MHD turbulence. Such a two- scale model is appropriate in solar wind in which the length scale is the order of the local heliocentric radial coordinate R. Fluctuations are broadband but generally have correlation scales, L_u and L_B, that are much smaller than R, at least for heliocentric distances of the order of 1 AU or more. Much of MHD turbulence activity takes place at inertial range scales that extend roughly form L_u down to scales 1000 times or so smaller, near the thermal ion gyroscale [20]. Thus the MHD turbulence activity of interest is well separated in length scale from the large-scale solar wind inhomogeneities. Moreover, the average large-scale properties are relatively reproducible, a property that allows description of canonical average flow and magnetic properties . The scale separation is the starting point in any of the two-scale direct interaction approximation (TSDIA) of Yoshizawa and co-workers [21].

We will restrict ourselves to the random and locally homogeneous MHD turbulence, which is often invoked in astrophysical and space application so that the problems become tractable. The incompressible MHD model is written in terms of the fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$, includes a momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \tilde{p} + \frac{1}{4\pi\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \tag{5}$$

and a magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \mu \nabla^2 \mathbf{B}. \tag{6}$$

The plasma density ρ, the kinematic viscosity ν, and the magnetic diffusivity μ, are assumed to be uniform constants. The velocity and magnetic field are solenoidal, $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0$ and the pressure $\tilde{p}$ is determined by taking the divergence of Eq. (5).

One of the critical issues that still need to be addressed in MHD turbulence is making refinement on the formulas for the viscosity, resistivity and other transport coefficients such as thermal conductivity. The classical work of Braginskii [22] [23] is still in use and progress in this area is highly desirable. Here we do not take a position on how viscosity and resistivity should be calculated in a particular application, but rather consider some implications of non- equipartition that could be induced by various agencies, such as non-unit magnetic Prandtl number.

Homogeneous MHD turbulence at high Reynolds number can be characterized by several different range of spatial scales:

1. The energy containing scales: The motions of these scales (L_u and L_B) control the overall dynamics of turbulence, and are directly responsible for turbulence energy transport processes.

2. The inertial range: The classical Kolmogorov theory [24] assumes that in the inertial range, the dynamics at an intermediate scale, r, cannot be influenced by the outer, low frequency scales, L_u and L_B, which are the scales of most external energy sources and forcing, nor can it be influenced by the inner, high frequency, viscous dissipation and magnetic diffusivity scales (η_u and η_B). This decoupling of the dynamics is the fundamental reason for the universal nature of the inertial range for a fully developed turbulence, regardless whether the spectra take the isotropic or anisotropic forms.

3. The dissipation range: The turbulent energy is transferred through the inertial range spectrum to the dissipation range where the dissipation process takes place. The onset of the dissipation range is marked by a sudden change in the spectral index (for example, the dissipation range for interplanetary magnetic field is detected [25] by observing a spectral exponent change from -1.703 to -4.228).

4. The far-dissipation range. The energy spectrum decreases exponentially with wavenumber k in hydrodynamic turbulence (see review by Zhou and Speziale [26]) .

III. TRANSITION TO FULLY-DEVELOPED TURBULENCE MHD: I. TAYLOR CONSTANT FLUX

In all MHD phenomenological or closure based theories, the existence of powerlaw scalings are assumed for both velocity and magnetic inertial ranges. The validity of these theories, in turn, are often assessed using data obtained from either simulations or observations. A fit to a powerlaw is a standard way to compare and contrast between the theory and data. However, it is often not clear whether the powerlaw fit is sufficient long to justify the existence of an inertial range.

Here, we develop a framework to provide two improved markers for a transition to turbulence in a MHD flow. These methodologies are based on the concepts of constant flux in an inertial range (this section) and of the separation of scales between the large and small scale boundary of the inertial range (next section).

A. Taylor's treatment of fluid turbulence

In fluid turbulence, Taylor [27] [28] proposed the kinetic-dissipation scaling, which has been a cornerstone of turbulence theory [24]. Batchelor [29] attempted to direct compare this formulation against experiments by plotting

$$D = \frac{L\epsilon}{\tilde{u}^3} \tag{7}$$

with

$$\epsilon = \frac{\nu}{2} < [\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]^2 > \tag{8}$$

and concluded, in the so-called initial period of decay, the data are not generally inconsistent with the formulation. Here the angle bracket denotes an appropriate ensemble average, or volume average. Sreenivasan [30] compiled relevant data from experiments and noted that D depends on some details of turbulence generation in experiments. As commented by Sreenivasan, investigators who compare theories of isotropic turbulence with grid turbulence data often implicitly assume that the turbulence sufficiently far behind a grid attains a character independent of the configuration of the grid. It does not quite appear justified, presumably because the scales of turbulence strongly affected by grid geometry contain a significant fraction of energy. Sreenivasan [31] and Kaneda *et al.* [32] also inspected direct numerical simulations in a periodic box and found that the numerical value of the constant appears to depend on details of forcing at low wavenumbers, or perhaps, the structure of the large-scale itself. All of these statements are consistent with the principle of self preservation introduced by von Karman and Howarth [33]. We now recall the principle of self preservation briefly [29]. For an isotropic turbulence to be self-preserving, the two point double and triple longitudinal velocity correlations, denoted by $f(r,t)$ and $g(r,t)$, must have

$$f(r,t) = f^*(r/L) \tag{9}$$

and

$$g(r,t) = g^*(r/L) \tag{10}$$

where $L = L(t)$ is a uniquely specified similarity length scale.

The principal message of both work is that D asymptotes to a constant value for Reynolds number based on Taylor microscale, Re_λ, is beyond 100 (or $Re \approx 10^4$, note that $Re_\lambda \approx Re^{1/2}$; see van Atta and Antonia [34]) which appears to be a necessary condition for fully developed turbulence. The Reynolds number based on Taylor microscale can be introduced

$$Re_\lambda = \frac{\tilde{u}\lambda}{\nu} \tag{11}$$

where the Taylor microscale λ is defined by

$$\epsilon \propto \nu \frac{\tilde{u}^2}{\lambda^2}. \tag{12}$$

We note that the Taylor microscale is directly related to the integral length scale through the Reynolds number. Noting

$$\frac{\tilde{u}^3}{L} = \nu \frac{\tilde{u}^2}{\lambda^2}, \tag{13}$$

we have

$$\lambda^2 = \frac{\nu L}{\tilde{u}}. \tag{14}$$

Using the definition of Re, we have

$$\lambda \propto Re^{-1/2} L. \tag{15}$$

B. Characterization of general MHD turbulence

We first decompose the velocity u_i and B_i into the mean fields $< u_i >$ and $< B_i >$ and the fluctuations u_i' and B_i'. We then follow the procedure outlined in the Chapter 3 of Tennekes and Lumley [35] to motivate our extending the results of previous section to MHD equations. The equation governing the kinetic energy $< u_i u_j > /2$ of the fluctuating velocity field is obtained by the momentum equation. In the same fashion, the equation governing the magnetic energy $< b_i b_j > /2$ of the fluctuating magnetic field can be constructed from the magnetic conducting equation. In the case of homogeneous shear flow, the rate of production of turbulent energy by Reynolds stress is balanced by the rate of viscous dissipation

$$\mathcal{P}_u = \epsilon_u, \quad \mathcal{P}_B = \epsilon_B. \tag{16}$$

where

$$\mathcal{P}_u =< u'_i u'_j > \frac{\partial < u_i >}{\partial x_j} + < u'_i b'_j > \frac{\partial < B_i >}{\partial x_j} \tag{17}$$

and

$$\mathcal{P}_B =< b'_i u'_j > \frac{\partial < B_i >}{\partial x_j} + < b'_i b'_j > \frac{\partial < u_i >}{\partial x_j}. \tag{18}$$

The estimation by Tennekes and Lumley resulted that

$$- < u'_i u'_j > \sim \tilde{u}^2, \quad \partial < u_i > / \partial x_j \sim \tilde{u}/L_u. \tag{19}$$

and

$$- < b'_i u'_j > \sim \tilde{u}\tilde{B}, \quad - < b'_i b'_j > \sim \tilde{B}^2, \quad \partial < b_i > / \partial x_j \sim \tilde{B}/L_B. \tag{20}$$

Based on above analysis, the MHD extension of the velocity dissipation rate, following the concept of Taylor, is then

$$\epsilon_u = D_u \frac{\tilde{u}^3 + \tilde{u}\tilde{B}^2}{L_u}. \tag{21}$$

The total kinetic energy is given by

$$K_u \equiv \frac{3}{2}\tilde{u}^2 = \int_0^\infty E_u(k) dk. \tag{22}$$

Note that the factor of 3/2 is resulted from the definition which makes the integral of the three dimensional spectrum $E_u(k)$ equal to the kinetic energy per unit mass (see Tennekes and Lumley [35], p. 251)

Again following Taylor, the magnetic dissipation rate is given

$$\epsilon_B = D_B \frac{\tilde{u}\tilde{B}^2}{L_B}. \tag{23}$$

The total magnetic energy is given by

$$K_B \equiv \frac{3}{2}\tilde{B}^2 = \int_0^\infty E_B(k)dk. \tag{24}$$

An ordering parameter will find use in next subsections

$$O_L = \frac{L_u}{L_B}, \tag{25}$$

where

$$L_u = \frac{\pi}{2\tilde{u}^2}\int_0^\infty \frac{E_u(k)}{k}dk \tag{26}$$

and

$$L_B = \frac{\pi}{2\tilde{B}^2}\int_0^\infty \frac{E_B(k)}{k}dk. \tag{27}$$

The case of $O_L \neq 1$ is relevant to astrophysical applications, such as that of magnetic amplification in galaxies (Son [15]).

Another interesting parameter is

$$O = \frac{\tilde{u}}{\tilde{B}} = \sqrt{\frac{K_u}{K_B}}. \tag{28}$$

A new type of Reynolds number will be needed

$$Rm^* = \frac{L_B\tilde{B}}{\mu}. \tag{29}$$

C. MHD Transition to fully-developed turbulence based on Taylor constant flux

A relationship between the MHD velocity field Taylor-microscale and the integral length can be established. For such velocity field part of MHD, we extend the treatment of Taylor,

$$\frac{\tilde{u}^3 + \tilde{u}\tilde{B}^2}{L_u} \propto \nu \frac{\tilde{u}^2}{\lambda_u^2}. \tag{30}$$

We can use the definitions and rearrange the above equation

$$\lambda_u^2 \propto \nu L_u \frac{\tilde{u}^2}{\tilde{u}^3 + \tilde{u}\tilde{B}^2} = L_u^2 \frac{1}{Re + O^{-1}Pr^{-1}O_L Rm^*}. \tag{31}$$

Therefore,

$$\lambda_u = C_1 L_u \frac{1}{(Re + O^{-1}Pr^{-1}O_L Rm^*)^{1/2}} \tag{32}$$

where C_1 is an order one constant.

The same procedure can be applied to the magnetic conducting equation part of MHD

$$\frac{\tilde{u}\tilde{B}^2}{L_B} \propto \mu \frac{\tilde{B}^2}{\lambda_B^2}. \tag{33}$$

We have

$$\lambda_B^2 \propto \mu L_B \frac{\tilde{B}^2}{\tilde{u}\tilde{B}^2} \tag{34}$$

which leads to

$$\lambda_B = C_2 L_B O_L^{1/2} Rm^{-1/2} \tag{35}$$

again, C_2 is an order one constant.

We remark here that in contrast to the case of fluid turbulence, for the general case of MHD turbulence, the velocity and magnetic Taylor microscales, Eqs. 32 and 35, depend on a number of other important nondimensional parameters.

The magnetic Reynolds number based on Taylor miscroscale can be introduced

$$Rm_\lambda = \frac{\tilde{u}\lambda_u}{\mu}. \tag{36}$$

An alternative Reynolds number may be of interest

$$Rm_\lambda^* = \frac{\tilde{B}\lambda_B}{\mu}. \tag{37}$$

We conjecture that the measurement

$$D_u = \frac{\epsilon_u L_u}{\tilde{u}^3 + \tilde{u}\tilde{B}^2} \tag{38}$$

will asymptote to a constant for newly defined Taylor based velocity Reynolds numbers above, perhaps 100 or above. This would be the necessary condition for the velocity flow to transition to a fully developed turbulence.

Note that the dissipation rate of velocity field is

$$\epsilon_u = \frac{\nu}{2} < [\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]^2 >= \int_0^\infty 2\nu k^2 E_u(k) dk, \tag{39}$$

as it was in the hydrodynamic case, while the magnetic dissipation rate is

$$\epsilon_B = \frac{\mu}{2} < [\frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i}]^2 >= \int_0^\infty 2\mu k^2 E_B(k) dk. \tag{40}$$

We further conjecture that the measurement

$$D_B = \frac{\epsilon_B L_B}{\tilde{u}\tilde{B}^2} \tag{41}$$

will asymptote to a constant for newly defined Taylor based magnetic Reynolds numbers above, perhaps 100 or above. This would be the necessary condition for the magnetic field to transition to a fully developed turbulence.

Instead of dual transition to fully-developed turbulence previously discussed, two distinctive single channel transitions may take place

1. *Fully developed velocity flow only*
2. *Fully developed magnetic field only*

Should such a transition takes place, the indicator would be that the corresponding constant, D_u or D_B, will become independent of the Reynolds number. For some of astrophysical applications discussed below, this type of single channel transition may take place. It also may be numerical simulated as the capacity of supercomputers advances rapidly in the future.

Politano and Poquet [36] [37] reported that the 4/5 law (The 4/3 scaling of the structure function is equivalent to the $-5/3$ Kolmogorov energy spectrum in wavenumber space) has the form u^3/L but evaluated in the inertial range. so for all those decay relations like Eqs. (38), and (41), the form that is taken may be consistent with what appears in the 4/5 law treatments.

IV. MHD TRANSITION TO FULLY-DEVELOPED TURBULENCE: II. KOMOGOROV SCALE SEPARATION

A. Scale separation in fluid turbulence

According to Kolmogorov, the inertial range exists where there is a scale that is free from both the external agencies at large-scale and from the dissipation processes in small-scale.

In the limiting case of hydrodynamic turbulence, the bounds of the velocity inertial range has been estimated by Dimotakis [38]. He found that the upper bound of inertial range is $L_{L-T} \approx 5.0 Re^{-1/2} L$, which is related to the Taylor microscale,λ, by the following relationship $L_{L-T} \approx 2.7\lambda$. The lower bound of the velocity inertial range (termed the inner viscous scale in Dimotakis): $L_\nu \approx 50 Re^{-3/4} L = 50\eta$.

From these estimates, the establishment of the inertial range, as illustrated by the emerging separation of large- and dissipation scales, is achieved when the condition

$$\frac{L_{L-T}}{L_\nu} > 1 \tag{42}$$

is met. With some margin, this condition is given

$$Re > 10^4. \tag{43}$$

The nice feature of above analysis is that a single condition, based on Re along, can be obtained as the indicator of transition to turbulence for fluid turbulence. The integral length scale, and other parameters, drop out of the final result. The key is that the transition occurs at a similar values of the Reynolds number for a wide range of flows. The fact that this transition is universal, i.e. independent of the flow geometry, would suggest that the same physical mechanism(s) apply [38].

We will show that, for our general MHD turbulence where $L_u \neq L_B$, and $Pr \neq 1$, more complicated conditions are required.

B. Inertial range for MHD flow

For MHD calculation, it is appropriate to argue that the necessary condition for the magnetic inertial range to begin to apply is the existence of a range of scales that are uncoupled from the large-scales velocity and magnetic field, on the one hand, and free from the effects of viscosity and magnetic diffusivity, on the other hand.

The classical Kolmogorov condition for existence of the inertial range is

$$\eta_u \ll r \ll L_u \tag{44}$$

and

$$\eta_B \ll r \ll L_B \tag{45}$$

for fully developed MHD.

As discussed in Tennekes and Lumley [35] (p.19), since the small-scale motions described by the momentum equation (Eq. 5) tend to have small time scales, one may assume that these motions are statistically independent of the relatively slow large-scale turbulence and of the external forcing. If this assumption makes sense, the small-scale motions should depend only on the rate at which it is supplied with energy by the large-scale motion and on the kinematic viscosity. It is fair to assume that the rate of energy supply should be equal to the rate of dissipation. This discussion suggests that the parameters governing the small-scale motion include at least the dissipation per unit mass ϵ_u (m^2sec^{-3}) and the kinematic viscosity ν (m^2sec^{-1}). This argument can be extended to the small scale magnetic fields governed by the magnetic conduction equation (Eq. 6). It is reasonable to assume that the small-scale magnetic fields activities should depend only on ϵ_B and μ. Hence, the velocity dissipation and magnetic diffusivity length scales are

$$\eta_u \equiv \left(\frac{\nu^3}{\epsilon_u}\right)^{1/4} \tag{46}$$

and

$$\eta_B \equiv \left(\frac{\mu^3}{\epsilon_B}\right)^{1/4} \tag{47}$$

for fully developed MHD.

Here, we would like to extend the same analysis of fluid turbulence to our general MHD turbulence. Our goal is to obtain a more precise definition on whether the velocity flow and/or magnetic field has transitioned to fully developed turbulence. The first order of business is to related the velocity dissipation and magnetic diffusivity length scales to the integral length scales, L_u and L_B, as well as other non-dimensional parameters, such as Pr, Re, Rm and Rm^*. The key step in that direction is to substitute Eqs. (21) and (23) into above definitions for η_u and η_B.

For the first equation, we have

$$\eta_u^4 = \frac{\nu^3}{\epsilon_u} = \frac{\nu^3 L_u}{D_u[\tilde{u}^3 + \tilde{u}\tilde{B}^2)]} = \frac{L_u^4}{D_u}\frac{1}{[Re^3 + \tilde{u}\tilde{B}^2(L_u^3/\nu^3)]}$$

As a result, we find that

$$\eta_u = D_u^{-1/4} L_u (Re^3 + O_L^2 Pr^{-2} Re Rm^*)^{-1/4} \tag{48}$$

and

We can carry out the same analysis for η_B

$$\eta_B^4 = \frac{\mu^3}{\epsilon_B} = \frac{\mu^3 L_B}{D_B \tilde{u}\tilde{B}^2} = \frac{L_u L_B^3}{D_B}\frac{1}{Re Pr \tilde{B}^2 (L_B^2/\mu^2)}.$$

This leads to

$$\eta_B = D_B^{-1/4} L_B O_L^{1/4} Re^{-1/4} Rm^{*-1/2} Pr^{-1/4}. \tag{49}$$

The range of the inertial range can be made more precise as follows. The large scale boundary of the velocity inertial range should be larger than the velocity or magnetic Taylor microscales but also should be smaller than the outer scale in order to free from the external agency. The small scale boundary of the inertial range should be larger than the Kolmogorov dissipation length scale, so it is not affected by the dissipation process.

The bounds of the magnetic initial range, at the moment, can be provisionally fixed. Let us extend the result of Dimotakis to MHD turbulence. We have

$$L_{u,L-T} \approx 2.17\lambda_u, \qquad L_{B,L-T} \approx 2.17\lambda_B, \tag{50}$$

where λ_u and λ_B are given in Eqs. (32) and (35), respectively.

The inner viscous scaled

$$L_\nu \approx 50\eta_u, \qquad L_\mu \approx 50\eta_B, \tag{51}$$

where η_u and η_B can be found in Eqs. (48) and (49).

We suggest that, for a transition to fully developed turbulence in the velocity field, requires

$$\frac{L_{u,L-T}}{L_\nu} > 1 \tag{52}$$

and

$$\frac{L_{B,L-T}}{L_\mu} > 1. \tag{53}$$

Above two equation can be written in terms of nondimensional parameters.

These two conditions are the extension of MHD turbulence from fluid turbulence. Again, as we pointed out when discussing the MHD extension of Taylor's constant flux, instead of dual transition, we may have two distinctive single channel transitions.

V. TIME SCALE, INTERACTING SCALES, AND ANISOTROPY

In the preceding sections we developed a phenomenology for non-equipartitioned MHD turbulence but we made little or no reference to the detailed structure of the cascade driven by the large scale dynamics, other than to estimate the small (Taylor and dissipation) length scales. However, MHD cascade do allow the structure and in particular may be highly anisotropic. Here we review some feature of the development of anisotropy with a view toward understand how it impacts the above results.

A spectral transfer timescale is estimated, incorporating effects due to both nonlinear straining motions

$$\tau_{nl} = \frac{1}{ku_k},\ u_k = [kE(k)]^{1/2} \tag{54}$$

and the sweeping-like influence of wave propagation

$$\tau_A = \frac{1}{kV_A},\ V_A = \frac{B_0}{\sqrt{4\pi\rho.}} \tag{55}$$

For equipartitioned MHD turbulence one may safely use the total (magnetic plus kinetic) energy spectrum in evaluating τ_{nl} and only a small uncertainty is introduced by this ambiguity in the estimate of straining-type nonlinear effects. We revisit this below.

The relative influence of strain and propagation effects will be closely related to the degree and type of anisotropy expected, e.g., whether anisotropy is relative to a strong externally supported DC magnetic field, or is relative to the local magnetic field. Accordingly, spectral transfer is either isotropic, when large samples of plasma are considered, or is anisotropic, when there is a strong large scale mean field. If the sweeping relative to the local magnetic field is not strong enough to displace straining as the prevailing local decorrelation, the strain will be dominated and the energy transfer will be local. This situation may be realized even if there is moderately strong local anisotropy relative to the local magnetic field direction. One reason for this is that the local mean field direction will be, in general, nonsteady, and will fluctuate on a nonlinear time scale. Consequently, nonlinear couplings can destroy the coherence of the local Alfvénic wave propagation effects in some cases. On the other hand, the DC or large-scale magnetic field imposes a preferred direction in anisotropic MHD turbulence. The sweeping effect is strong and energy transfer is suppressed in the parallel direction. The fundamental reason for this is that wave propagation effects decorrelate the nonlinear couplings associated with spectral transfer parallel to the mean magnetic field; however, spectral transfer towards higher perpendicular wavenumber is uninhibited.

In the past, much has been made of differences between Kolmogorov [39] [40] and IK (Iroshnikov & Kraichnan) [41] [42] spectra. In interplanetary observations and numerical simulations, these two cases are sometimes compared to attempt to draw a sharp distinction. In our current perspective, the MHD turbulence experiences a smooth variations between such different limits when the time scale for decaying of the transfer correlation function is varied. Observation of a prevalent spectral index reflects relative dominance of sweeping effects versus straining effects, or equivalently local effects versus nonlocal effects. (Zhou, Matthaeus, and Dmitruk [43])

A. Isotropic

The starting point for estimating the energy spectrum, $E(k)$, in the various regimes of MHD turbulence is the steady state phenomenological energy transfer flux, ϵ,

$$\epsilon = \tau_T(k)\frac{kE(k)}{(\tau_{nl}^{\pm})^2}. \tag{56}$$

Matthaeus and Zhou [44] and Zhou and Matthaeus [45] proposed that the lifetime of transfer function correlation

$$\frac{1}{\tau_T(k)} = \frac{1}{\tau_{nl}(k)} + \frac{1}{\tau_A(k)}. \tag{57}$$

which incorporates the influence of both the external agent and turbulent nonlinear interactions. This leads to both the Kolmogorov -5/3 and IK -3/2 spectra when the straining and sweeping limits are achieved.

B. Anisotropy

Anisotropy is controlled by variation of the directed Alfvénic decorrelation time in comparison to the local nonlinear time scale. Suppose that there is considerable energy in 2D or quasi- 2D modes that satisfy

$$\tau_A(k_{||}) > \tau_{nl}(k). \tag{58}$$

The dominant features of those of 2D MHD turbulence, or quasi-2D turbulence, equivalent to the "mean modes" of Reduced MHD turbulence (Kinney and McWilliams [46]). Two limiting possibilities can arise when the anisotropic MHD is dominated by resonant interactions. The quasi-2D turbulence experience decay of triple correlations mainly due to propagation effects in the 2D plane, due to the large scale 2D magnetic field. Alternatively, the quasi-2D motions can be governed by strain. The corresponding time scales are

$$\tau_T(k_\perp) = \frac{1}{k_\perp \delta v_A} \tag{59}$$

where δv_A is the Alfvén speed associated with the large-scale quasi-2D magnetic field; and

$$\tau_{nl}(k_\perp) = k_\perp^{-3/2} E^{-1/2}(k_\perp). \tag{60}$$

In the case of anisotropic MHD turbulence is when the quasi-2D modes are not strong enough to control the decorrelation of the more wave-like modes. In the so-called weak MHD turbulence situation, neither the quasi-2D cascade itself, nor the resonant cascade it induces is the dominant feature that determine the non-quasi-2D waves. For such as case, the transfer correlations should decay due to higher order wave propagation processes. Ng and Bhattacharjee [47] and Galtier *et al.* [48] have modified the Kraichnan [42] argument by taking into account the anisotropic feature in the characteristic Alfvén time scales

$$\tau_A(k_{||}) \sim 1/[V_A k_{||}]. \tag{61}$$

The energy flux in the inertial range can be determined by

$$\epsilon = \tau_T(k_\perp, k_{||}) \frac{k_\perp E(k_\perp)}{(\tau_{nl}(k_\perp))^2} \tag{62}$$

where

$$\frac{1}{\tau_T(k_\perp, k_{||})} = \frac{1}{\tau_{nl}(k_\perp)} + \frac{1}{\tau_A(k_\perp)} + \frac{1}{\tau_A(k_{||})}. \tag{63}$$

These two equations will lead to the anisotropic energy spectrum $E(k_\perp)$ changes smoothly in spectral $k_\perp$ space with the indices varying between -5/3 , -3/2, and -2, taking on these specific values in cases for which the first, second or third terms on the right hand side, respectively, become the dominant contribution to the so called triple decay rate $1/\tau_T$.

As a result, a wide range of possibilities for the spectral anisotropy and for spectral indices can be emerged in the general MHD framework. With a strong uniform magnetic field, the resulting anisotropic energy spectrum can reduce to $k_\perp^{-5/3}$ when resonant interactions and quasi-two dimensional strain is the dominant decorrelation effect, or to $k_\perp^{-3/2}$ when large scale quasi-2D Alfvénic decorrelation is strong. When quasi-2D effects are weak, the spectrum can become either $k_\perp^{-2}$ ("weak turbulence") when local interactions are dominant, or $k_\perp^{-3}$ when nonlocal interactions are dominant. When both local and nonlocal interactions are present, the spectrum varies smoothly between these limits.

The implications of the physics of spectral anisotropy are still being explored and applied in various astrophysical settings, The potential to achieve an even greater range of possibilities for MHD turbulence spectra emerges when upon realizing that the the crucial nonlinear time scale as given in Eq. (54) can be readily generalized along the lines of Section II above. Consequently, one might consider separately the the dynamical balance between velocity field induced strain and propagation ' effects, and magnetic field induced strain and propagation effects. These may differ greatly for nonequipartitioned cases since for example, $\tau_{nl} = \frac{1}{ku_k}$ and $\tau_{nl,b} \equiv \frac{1}{kb_k}$, $b_k = \sqrt{kE_b(k)}$, may greatly differ. We defer to a later time an examination of these physically distinct cases.

VI. DISCUSSION AND CONCLUSION

In order to understand the detailed structure of the energy spectrum $E(k_\perp, k_\parallel)$ one needs to analyze spectral transfer in great detail. Both local and nonlocal interactions [49–51,26] can be involved, and the balance of different relaxation rate in Eq. (63) may vary considerably on different part of k space. As an example of an approximate treatment, Goldreich and Sridhar [52,53] retained two of the effects in 63, $\tau_{nl}(k_\perp)$ and $\tau_{k_\parallel}$. (n.b. By ignoring $\tau_A(k_\perp)$ they rule out the Kraichnan spectrum $k_\perp^{-3/2}$.) Subsequently, assuming a steady state, Goldreich and Sridhar found the locus of positions in $\mathbf{k}$ space for which the triple decay rate has equal contribution for nonlocal strain and wave propagating effects, under the assumption of steady perpendicular spectral transfer. This gives rise, in that special case, to the so-called "critical balance" boundary for a steady anisotropic Kolmogorov cascade. Other possibilities exists that are more general, for example in nonsteady cases, for cases in which purely perpendicular transfer is not a good approximation, or in the energy containing range. With regard to energy decay and identification of candidates for strong turbulence, few of these details are essential.

It is crucial, however, to realize that in the anisotropic case, the cascade is driven by perpendicular gradient. Therefore, the length scales, such as L_u and L_B, for example, should be interpreted as perpendicular length scales. In addition, the cascade model typically give prediction for the total energy spectrum $E(k)$ alone. The distribution of energy into the magnetic $E_B(k)$ and kinetic $E_u(k)$ components is an additional interesting issue. However, in the discussion of transition to of strong turbulence and the decay of energy as Sections 3-4 above. such detailed information is not required.

We have focused on developing a phenomenology for non-equal partition MHD flow, which allows disparity magnetic Prandtl number, dissimilar kinetic and magnetic Reynolds number, different outer kinetic and magnetic length scales, and strong anisotropy.

We established two conditions for transition to fully developed MHD turbulence based on fundamental concepts of

(i) Taylor's constant flux, and

(ii) Kolmogorov's scale separation.

For this analysis, the detailed turbulence structure is not needed. These two conditions for MHD transition are expected to provide consistent predictions and should be applicable to anisotropic MHD flows, after the length scales are replaced by their corresponding perpendicular components.

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REFERENCES

[1] T.G. Cowling, Magnetohydrodynamics, (Interscience, New York, 1957)

[2] J.A. Shercliff, A text book of magnetohydrodynamics (Pergamon Press, Oxford, 1965)

[3] D. Biskamp, Magnetohydrodynamic turbulence, (New York, Cambridge Univ Press, 2003)

[4] S.R. Cranmer and A.A. van Ballegooijen, Astrophys. J., **594**, 573 (2001)

[5] M.L. Goldstein, D.A. Roberts, and W.H. Matthaeus, Annu. Rev. Astron. Astrophys., **33**, 283 (1995)

[6] C.Y. Tu and E. Marsch, Space Sci. Rev., **73**, 1 (1995)

[7] M.L. Goldstein and D.A. Roberts, Phys. Plasma, **8**, 4154 (1999)

[8] W.H. Matthaeus and M.L. Goldstein, J. Geophys. Res., **87**, 6011 (1982)

[9] A.A. Schekochihin, J.L. Maron, S.C. Cowley, and J.C. McWilliam, Astrophys. J., **576**, 806 (2002)

[10] R.J. West, S. Nazarenko, J.-P. Laval, and S. Galtier, Astron. Astrophys., **414**, 807 (2004)

[11] J. Maron, S. Cowley, and J. McWilliams, Astrophys. J., **603**, 569 (2004)

[12] M. Christensson, M. Hindmarsh, and A. Brandenburg, Phys. Rev. E., **64**, 056405 (2001)

[13] A. Pouquet, U. Frisch, and J. Leorat, J. Fluid Mech., **77**, 321 (1976)

[14] Leorat, A. Pouquet, and U. Frisch, J. Fluid Mech. **104**, 419 (1981)

[15] D.T. Son, Phys. Rev. D, **59**, 06308 (1999)

[16] T.A. Yousef, A. Brandenburg, and G. Rdiger, Astron. Astrophys., **411**, 321 (2003)

[17] S.A. Balbus and J.F. Hawley, Rev. Mod. Phys., **70**, 1 (1998)

[18] M-M Mac Low and R.S. Klessen, Rev. Mod. Phys.,**76**, 125 (2004)

[19] Y. Zhou and W.H. Matthaeus, J. Geophys. Res. **95**, 10291 (1990)

[20] A.J. Hundausen, Coronal expansion and solar wind, (Springer-Verlag, New York, 172)

[21] A. Yoshizawa, K. Itoh, and I. Itoh, Plasma and fluid turbulence: theory and modeling, (Institute of Physics, Bristol, 2003)

[22] S.I. Braginskii, *in* Reviews of Plasma Phys. Vol. , (Consultants Bureau, New York, 1965)

[23] NRL Plasma Formulary, edited by J.D. Huba (Naval Research Laboratory, Washington, D.C., 2000)

[24] A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics (MIT Press), Vol. 2 (1975).

[25] R.J. Leamon, C.W. Smith, N.F. Ness, W.H. Matthaeus, and H.K. Wong, J. Geophys. Res., **103**, 4775 (1998)

[26] Y. Zhou and C.G. Speziale, Applied Mech. Rev., **51**, 267. (1998)

[27] G.I. Taylor, Proc. R. Soc. Lond. A, **151**, 422 (1935)

[28] G.I. Taylor, J. Aeronaut. Sci. **IV**, 31 (1937)

[29] G. K. Batchelor, Theory of Homogeneous Turbulence, (Cambridge University Press, New York, 1953).

[30] K.R. Sreenivasan, Phys. Fluids,**27**, 1048 (1984)

[31] K.R. Sreenivasan, Phys. Fluids, **10**, 528 (1998)

[32] Y. Kaneda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, Phys. Fluids, **15**, L21 (2003)

[33] T. von Kármán and L. Howarth, Proc. Roy. Soc. London Ser. A **164**, 192 (1938).

[34] C.W. van Atta and R.A. Antonia, Phys. Fluids, **23**, 252 (1980)

[35] H. Tennekes and J.L. Lumley, A First Course in Turbulence, (MIT Press, Cambridge, MA 1972)

[36] H. Politano and A. Pouquet, Phys. Rev. E., **57**, R21 (1998)

[37] H. Politano and A. Pouquet, Geophys. Res. Lett., **25**, 273 (1998)

[38] P.E. Dimotakis, J. Fluid Mech. **409**, 69 (2000)

[39] D. D. Fyfe and D. Montgomery, J. Plasma Phys., **16**, 181 (1976)

[40] D. D. Fyfe, D. Montgomery, and G. Joyce, J. Plasma Phys., **17**, 369 (1977)

[41] P. Iroshnikov, Soviet Astron., **7**, 566 (1963)

[42] R.H. Kraichnan, Phys. Fluids, **8**, 1385 (1965)

[43] Y. Zhou, W.H. Matthaeus, and P. Dmitruk, Rev. Mod. Phys., **76**, 1015 (2004)

[44] W.H. Matthaeus and Y. Zhou, Phys. Fluids B, **1**, 1929 (1989)

[45] Y. Zhou and W.H. Matthaeus, J. Geophys. Res., **95**, 14881 (1990)

[46] R.M. Kinney, and J.C. McWilliams, Phys. Rev. E, **57**, 7111 (1998)

[47] C.S. Ng, and A. Bhattacharjee, Phys. Plasmas, **4**, 605 (1997)

[48] S. Galtier, S.V. Nazarenko, A.C. Newell, and A. Pouquet, J. Plasma Phys., **63**, 447 (2000)

[49] Y. Zhou, Phys. Fluids A, **5**, 1092 (1993)

[50] Y. Zhou, Phys. Fluids A, **5**, 2511 (1993)

[51] Y. Zhou, P.K. Yeung, and J.G. Brasseur, Phys. Rev. E, **53**, 1261 (1996)

[52] P. Goldreich and S. Sridhar, Astrophys. J., **438**, 763 (1995)

[53] P. Goldreich and S. Sridhar, Astrophys. J., **485**, 680 (1997)